Laplace & Inverse Laplace Transforms

Integral Transforms \[ \mathcal{L} \left[ f(t) \right] = \int_{0^-}^{\infty} f(t)e^{-st}dt \]
\[ \mathcal{L}^{-1} \left[ F(s) \right] = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds \]

- time domain \(\rightarrow\) frequency domain
- differential equation \(\rightarrow\) algebraic equation

Strategy for solving Linear Differential Equations:
1. Convert eqn in terms of \(f(t)\) to eqn in terms of \(F(s)\)
2. Solve for \(F(s)\).
3. Convert back via Partial Fraction Expansion.

Partial Fraction Expansion

Method for simplifying a ratio of two polynomials: \(N(s)/D(s)\).

Strategy:
1. Simplify \(F(s)\) s.t. order(\(N(s)\)) < order(\(D(s)\))
2. Factorize – based on the following 3 cases:
   1. Roots of \(D(s)\) are Real & Distinct
      \[ F(s) = \frac{1}{s^2 + 5s + 6} = \frac{A}{s + 2} + \frac{B}{s + 3} \]
   2. Roots of \(D(s)\) are Real & Repeated
      \[ F(s) = \frac{1}{(s^2 + 5s + 6)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3} \quad \frac{C}{(s + 3)^2} \]
   3. Roots of \(D(s)\) are Complex or Imaginary
      \[ F(s) = \frac{1}{(s^2 + s + 5)(s + 3)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + s + 5} \]

Underlying Assumptions

- Laplace Transforms are Linear Transformations
  - Linearity
  - Homogeneity

- As such, when solving ODE’s via Laplace Transforms
  - All initial conditions are assumed to be 0.
  - Coefficients in ODE must be time-invariant, i.e. not a function of \(t\)
  - Such systems are called: Linear Time-Invariant Systems

Transfer Functions

Given: \(c(t)\) – output, \(r(t)\) – input
\[ a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t) \]
\[ R(s) = \left( \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{G(s)} \right) \quad \text{C(s)} \]

- Relates output to input
- \(G(s)\) == Transfer Function == output/input
- Furthermore, given \(G(s)\) and \(R(s)\), \(C(s) = G(s)R(s)\)
- Then \(c(t) = \mathcal{L}^{-1} \left[ G(s) \ast \text{input} \right] \)
State Space Representation

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

- State Variables = order of the original differential equation
- Dimension of the state space = # of state variables
- Non-zero initial conditions
- Multiple inputs and outputs
- Nonlinear systems

Transfer Function → State Space

\[ \frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} \]
\[ R(s) \rightarrow \frac{24}{s^3 + 9s^2 + 26s + 24} \rightarrow C(s) \]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-24 & -26 & -9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
24
\end{bmatrix}
\]

\[ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} \]

Transfer Function → State Space

\[ R(s) \rightarrow \frac{b_3s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0} \rightarrow C(s) \]

\[ R(s) \rightarrow \frac{1}{a_3s^3 + a_2s^2 + a_1s + a_0} \rightarrow X(s) \rightarrow \frac{b_3s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0} \rightarrow C(s) \]

State Space → Transfer Function

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \]
Suppose given, \( \tilde{u}(t) \), we obtain the solution \( \tilde{x}(t) \). Consider

\[
\begin{align*}
    u(t) &= \tilde{u}(t) + \delta(t) \\
    x_0 &= \tilde{x}_0 + x_{00}
\end{align*}
\]

Assume, that \( x(t) = \tilde{x}(t) + x_\delta(t) \)

Summary

- State-Space Representation
- Transfer Function \( \Leftrightarrow \) State-Space Representation
- Linearization

\[
J \frac{d^2 \theta}{dt^2} + \frac{MgL}{2} \sin \theta = T
\]