From Last Time

• Block Diagrams

• Signal Flow Diagrams

• Mason’s Gain Rule/Formula

\[ G(s) = \frac{1}{\Delta} \sum_{k=1}^{p} T_k \Delta_k \]

Similarity Transformations

• Similar systems have the same transfer functions

• Transformations can be achieved WITHOUT transfer functions or signal flow graphs

• Given the system

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

• Assume, \( x = Pz \), then the above system can be rewritten as

\[ \dot{z} = P^{-1}APz + P^{-1}Bu \]
\[ y = CPx + Du \]

• \( P \) is the transformation matrix

Matrix Diagonalization

\[ Av_i = \lambda_i v_i \]

• Given an \( n \times n \) square matrix \( A \), \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors.

\[ P^{-1}AP = \Lambda \]
\[ A = P\Lambda P^{-1} \]

• If \( A \) is symmetric, then \( P \) is orthogonal

• A little careful …

Poles, Zeros, and System Response

From Differential Equation:

Output Response = Particular + Homogeneous

Forced/Steady-State Response

Natural Response

So far, solving differential equations:

• Brute Force

• Laplace Transform

Question:

• Can we get a qualitative (rather than exact) solution for system response?

• Why?
### Poles

Given a transfer function: \( G(s) = \frac{N(s)}{D(s)} \)

Definition:
\[ \{s \mid D(s) = 0 \text{ or } D(s) = 0 \text{ s.t. } N(s) = 0\} \]

Example:
\[ G(s) = \frac{s + 2}{s(s + 5)} \]

Conventionally, given
\[ G(s) = \frac{s^2 + 2s + 1}{s(s + 1)(s + 5)} \]

\( s = -1 \) is a pole, even though \( G(-1) \neq \infty \)

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### Zeros

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Definition:
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Example:
\[ G(s) = \frac{s + 2}{s(s + 5)} \]

Conventionally, given
\[ G(s) = \frac{(s + 1)(s + 2)}{s(s + 1)(s + 5)} \]

\( s = -1 \) is a zero, even though \( G(-1) \neq 0 \)

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### The s-plane

Recall, \( s = \sigma + j\omega \)

- **Zeros** – ◯
- **Poles** – x

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### First Order Systems

In general, given
\[ G(s) = \frac{s + b}{s + a} \]

Let \( R(s) = 1/s \), then
\[ C(s) = \frac{s + b}{s(s + a)} = \frac{K_1}{s} + \frac{K_2}{s + a} \]

As such,
\[ c(t) = K_1 + K_2e^{-at} \]

\[ K_1 = \frac{b}{a} \]

\[ K_2 = 1 - \frac{b}{a} \]

Therefore,
\[ c(t) = \frac{b}{a} + (1 - \frac{b}{a})e^{-at} \]
**First Order Systems**

Given, \( G(s) = \frac{s + b}{s + a} \) \( \) with \( R(s) = \frac{1}{s} \), results in

\[ c(t) = \frac{b}{a} + \left(1 - \frac{b}{a}\right)e^{-at} \]

Note the following:
1. Pole of \( R(s) \) generates the form of the **forced response**
2. Pole of \( G(s) \) generates the form of the **natural response**
3. Pole on the real axis generates an exponential response of the form \( e^{-\alpha t} \), where \( -\alpha \) is the pole location on the axis
4. Zeros and poles generate the amplitudes of the responses

**Obtaining \( G(s) \) Empirically**

- Assume \( G(s) = \frac{K}{s + a} \) has a step response of
  \[ C(s) = \frac{K}{s(s + a)} = \frac{K}{s} \frac{1}{s + a} \]
- If the signal looks like a 1st order response, then obtain \( K \) and \( a \) by measuring \( T_c \), \( T_r \), and \( T_s \)

**Characterizing First – Order Systems**

- **Time Constant:**
  - Time for \( e^{-\alpha t} \) to decay 37% of its initial value, \( T_c = 1/a \)
- **Rise Time:**
  - Time for the signal to go from 0.1 to 0.9 of its final value, \( T_r = 2.2/a \)
- **Settling Time:**
  - Time for the signal to reach & stay within 2% of its final value, \( T_s = 4/a \)

**Second – Order Systems**

Given, \( G(s) = \frac{1}{s^2 + bs + c} \) \( \) and \( R(s) = \frac{1}{s} \), we know

\[ C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{s + r_2} \]

Possible solutions
- \( r_1 \) and \( r_2 \) are real & distinct
- \( r_1 \) and \( r_2 \) are real & repeated
- \( r_1 \) and \( r_2 \) are both imaginary
- \( r_1 \) and \( r_2 \) are complex conjugates
Case 1: Roots are Real & Distinct
\[ C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{s + r_2} \]
This gives \[ c(t) = K_1 + K_2e^{-r_1t} + K_3e^{-r_2t} \]

Overdamped response

Case 2: Roots are Real and Repeated
\[ C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{(s + r_1)^2} \]
This gives \[ c(t) = K_1 + K_2e^{-r_1t} + K_3te^{-r_1t} \]

Critically damped response

Case 3: Both Roots Are Imaginary
\[ C(s) = \frac{1}{s(s^2 + \omega^2)} = \frac{K_1}{s} + \frac{K_2s + K_3}{s^2 + \omega^2} \]
Gives \[ c(t) = K_1 + K_4 \cos(\omega t - \phi) \]

Undamped response

Case 4: Roots Are Complex
\[ C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2s + K_3}{s^2 + bs + c} = \frac{K_1}{s} + \frac{K_4(s + \sigma)}{(s + \sigma)^2 + \omega^2} \]
Gives \[ c(t) = K_1 + K_4e^{-\sigma t} \cos(\omega t - \phi) \]

Underdamped response
A Closer Look at Case 4

Exponential decay generated by real part of complex pole pair

Sinusoidal oscillation generated by imaginary part of complex pole pair

Second – Order Systems Summary

Given, $G(s) = \frac{1}{s^2 + bs + c}$ and $R(s) = 1/s$

Solution is one of the following:
1. $r_1$ and $r_2$ are real & distinct => Overdamped Response
   - Two Poles @ $-\sigma_1, -\sigma_2$
2. $r_1$ and $r_2$ are real & repeated => Critically Damped Response
   - Two Poles @ $-\sigma$
3. $r_1$ and $r_2$ are both imaginary => Undamped Response
   - Two Imaginary Poles @ $\pm j\omega$
4. $r_1$ and $r_2$ are complex conjugates => Underdamped Response
   - Two Complex Poles @ $-\sigma \pm j\omega$

Characterizing 2nd Order Systems

Given, $G(s) = \frac{c}{s^2 + bs + c}$ and $R(s) = 1/s$, with $s = c + \omega_n^2$

• Natural Frequency ($\omega_n$):
  - frequency of oscillation of the system w/o damping

• Damping Ratio ($\zeta$):
  - $\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/sec)}} = \frac{1}{\text{Natural period (sec)}} = \frac{\omega_n}{2\pi \text{Exponential time constant}}$

• When $b = 0$, $G(s) = \frac{c}{s^2 + c}$, with $s = \pm j\sqrt{c}$, thus $\omega_n = \sqrt{c}$, with $c = \omega_n^2$

• Assuming an underdamped system, $s = -\sigma \pm j\omega_n$, w/ $\sigma = -b/2$, thus $\zeta = \frac{\omega_n}{\omega_n} = \frac{a/2}{\omega_n^2}$, $b = 2\zeta \omega_n$
General 2\textsuperscript{nd} Order Systems

Thus, in general, second-order transfer functions have the form

\[ G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \]

with poles of the form

\[ s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

Given,

\[ G(s) = \frac{36}{(s^2 + 4.2s + 36)} \]

Compute \( \zeta \), \( \omega_n \), and \( s_{1,2} \)?

The General Underdamped System

Given \( G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \) and \( R(s) = 1/s \) such that \( \zeta < 0 \) and \( \omega_n > 0 \), using PFE we obtain

\[ C(s) = \frac{K_1}{s} + \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2} \]

Taking the inverse Laplace results in

\[ c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos \left( \omega_n \sqrt{1-\zeta^2} t - \phi \right) \]

with \( \phi = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \)

Underdamped Responses

Characterizing Underdamped Systems

General solutions is of the form:

\[ c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos \left( \omega_n \sqrt{1-\zeta^2} t - \phi \right) \]

- Rise Time, \( T_r \)
  - Time required for the signal to go from 0.1 to 0.9 of the final value
- Peak Time, \( T_p \)
  - Time required to reach the maximum (or first) peak
- Percentage Overshoot, \%OS
  - Amount the signal overshoots the steady-state value at \( T_p \), expressed as % of steady-state
- Settling Time, \( T_s \)
  - Time required for damped oscillations to be within ±2% of steady-state
Characterizing Underdamped Systems

**Peak Time:**

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d} \]

\[ T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d} \]

\[ \%OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100\% \]

Evaulating \( T_p, \%OS, T_s, T_r \)

**Peak Time:**

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \]

\[ \%OS = e^{-\zeta \pi \sqrt{1 - \zeta^2}} \times 100\% \]

(Note: given \%OS, we can use the equation to solve for \( \zeta \))

**Settling Time:**

\[ T_s = \frac{4}{\zeta \omega_n} \]

**Rise Time:** obtained numerically

- As \( \omega_d \) increases, \( T_p \) decreases
Settling Time

- As $-\sigma_d$ increases, $T_s$ decreases

\[ T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d} \]

Overshoot

- Let $\theta$ be the angle made by $-\sigma_d + j\omega_d$
- $\sigma_d = \cos \theta$

\[ \%OS = e^{-\zeta \pi / \sqrt{1-\zeta^2}} \times 100\% \]

In Summary

Example

- %OS
- $T_p$ and $T_s$
- $K = 5 \text{ N-m/rad}$
- $J$, $D$
Higher Order Systems

- Approximation via 2nd order system?
  - Yes, only under certain conditions

Adding Zeros to 2nd Order Systems

Given $C(s)$, consider the effects of adding an extra zero @ $-a$

Equivalent to $(s+a)C(s) = sC(s) + aC(s)$

If $a$ is large, then …

If $-a < 0$, then

Laplace Transform Solution to State Equations

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

\[
\frac{Y(s)}{U(s)} = \frac{C \text{adj}(sI - A)B + D \text{det}(sI - A)}{\text{det}(sI - A)}
\]

Eigenvalues of $A \equiv$ Transfer Function Poles

Dominant Poles

$\alpha R \gg \zeta \omega_n$
Time Domain Solution of State Equations