From Last Time

Cayley – Hamilton Theorem:
Given a matrix \( A \), \( A \) satisfies its own characteristic polynomial.

- No reason to compute matrix powers higher than \( A^n \)
- Can express functions composed of matrix polynomials
- What about functions of \( A \) that are NOT polynomials?
  - Use Taylor series
  - Examples:
    \[
    \sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \ldots
    \]
    \[
    \cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \ldots
    \]
    \[
    e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots
    \]

Properties of \( e^{At} \)

- \( A e^{At} = e^{At} A \)
- Given \( A = P \Lambda P^{-1} \), then \( e^{At} = P e^{\Lambda t} P^{-1} \)
- Helps us to compute solutions of state equations

Solution of State Equation

Given \( \dot{x} = Ax + Bu \) with \( x(0) = x_0 \)
\[
\begin{align*}
y &= Cx + Du \\
\end{align*}
\]

Then starting with \( \dot{x} = Ax + Bu \)
\( (\dot{x} - Ax) = Bu \)

Multiply both sides by \( e^{-At} \), we get
\[
\frac{d}{dt} (e^{-At} x(t)) = e^{-At} Bu
\]

Integrating both sides, we obtain
\[
x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau
\]
Example 1

Given: \[ \dot{x} = \begin{bmatrix} 0 & 2 \\ -2 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \]

\[ y = [2 \ 1] x \]

Eigenvalues of A: -1, -4

\[ P = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \] \[ p^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \]

\[ e^{At} = Pe^{At}p^{-1} = \begin{bmatrix} 4e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{1}{3}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix} \]

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Stability

- Total response of the system is
  \[ c(t) = c_{forced}(t) + c_{natural}(t) \]

- A linear time-invariant (LTI) system is
  - Stable if \( c_{natural}(t) \to 0 \) as \( t \to \infty \)
  - Unstable if \( c_{natural}(t) \to \infty \) as \( t \to \infty \)
  - Marginally stable if \( c_{natural}(t) \) neither grows or decays as \( t \to \infty \)

- Another definition – Bounded Input Bounded Output (BIBO)
  - A system is stable if every bounded input yields a bounded output
  - A system is unstable if every bounded input yields an unbounded output

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\[ \delta - \varepsilon \text{ relationship of stability} \]

- Stability in the sense of Lyapunov
  - \( c(t) \) is stable if and only if for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that
  \[ \|c(0)\| < \delta \]
  \[ \Rightarrow \|c(t)\| < \varepsilon, \quad t > 0 \]

- Asymptotic Stability
  - \( c(t) \) is asymptotically stable if and only if for any \( \delta > 0 \) such that
  \[ \|c(0)\| < \delta \]
  \[ \Rightarrow \|c(t)\| \to 0, \quad t \to \infty \]
Stability & Location of Poles

A few more words about pole locations

• Equivalent systems
  • If poles are in left-half plane
    • $(s+a)$
    • Positive coefficients
    • All powers present

• Instability – Sufficient Condition
  • Signs of coefficients are not the same
  • Missing powers – (maybe marginally stable)

Routh-Hurwitz

• Provides stability information without requiring to explicitly solve for poles
• Trivial for analysis
• Provides bounds for design
• 2 Steps:
  • Step 1: Generate Routh table
  • Step 2: Interpret the Routh table following Routh-Hurwitz criterion

Generating a Basic Routh Table

<table>
<thead>
<tr>
<th>R(s)</th>
<th>N(s)</th>
<th>C(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$</td>
<td></td>
</tr>
<tr>
<td>$s^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2

![Routh Table Diagram]

Using Matlab we get:
-13.4136
1.7068 + j8.5950
1.7068 - j8.5950

What can you conclude about your Routh Table?

Routh-Hurwitz Criterion

- # of roots located in the RHP == # of sign changes in the 1st column of Routh table

<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
<th>31</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>10</td>
<td>1030</td>
<td>0</td>
</tr>
<tr>
<td>s^1</td>
<td>-72</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>103</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 3

D(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6

Using Matlab:
-2.2859
-1.0265 + j0.6541
-1.0265 - j0.6541
0.6404 + j0.7106
0.6404 - j0.7106
0.0291 + j0.8028
0.0291 - j0.8028

Example 4

\[ T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \]

<table>
<thead>
<tr>
<th>s^5</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^4</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>s^3</td>
<td>/ε</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Another Method to Handle 0s in 1st Column

- Fact:
  - $R_1 = \text{Roots(Polynomial 1)}$
  - $R_2 = \text{Roots(Polynomial 2)}$
  - For every $r_i \in R_1$ and $r_j \in R_2$, $r_i = 1/r_j$ for all $i = j = 1, \ldots, p$
  - Then # of $r_i$ in RHP & LHP == # of $r_j$ in RHP & LHP

- Given: $s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$

- Let $s = 1/d$, then

$$\left(\frac{1}{d}\right)^n \left[1 + \alpha_{n-1}d + \cdots + \alpha_1d^{n-1} + \alpha_0d^n\right] = 0$$

Example 4 – Again

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

$$D(d) = 3d^5 + 5d^4 + 6d^3 + 3d^2 + 2d + 1$$

<table>
<thead>
<tr>
<th>$d^5$</th>
<th>3</th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^4$</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$d^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 5

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

<table>
<thead>
<tr>
<th>$s^5$</th>
<th>1</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^4$</td>
<td>1</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>$s^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Entire Row in Routh Table is 0

- From our previous example, row right before the row of 0s is
  $$P(s) = s^5 + 6s^2 + 8$$

  then, compute $dP/ds = 4s^3 + 12s + 0$

- Replace the row of zeros with the coefficients of $dP/ds$

<table>
<thead>
<tr>
<th>$s^5$</th>
<th>1</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^4$</td>
<td>1</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>$s^3$</td>
<td>1</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>$s^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Why does this work?

- A row of zeros appear when a purely even or purely odd polynomial is a factor of D(s)
  - Ex: \( s^4 + 5s^2 + 7 \) – even
  - \( s^5 + 5s^3 + 7s + 1 \) – odd
- Even polynomials only have roots that are symmetrical about the origin

Furthermore

\[
T(s) = \frac{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}{20}
\]

| \( s^8 \) | 1 | 12 | 39 | 48 | 20 |
| \( s^7 \) | -1 | 22 | 59 | 38 | 0 |
| \( s^6 \) | 1 | 3 | 2 | 0 | 0 |
| \( s^5 \) | 1 | 3 | 2 | 0 | 0 |
| \( s^4 \) | 2 | 3 | 2 | 0 | 0 |
| \( s^3 \) | 3 | 4 | 0 | 0 | 0 |
| \( s^2 \) | 1/3 | 0 | 0 | 0 | 0 |
| \( s^1 \) | 4 | 0 | 0 | 0 | 0 |

Odd RHP Poles: 2
Odd LHP Poles: 2
Odd \( j\omega \) Poles: 0
Even RHP Poles: 0
Even LHP Poles: 0
Even \( j\omega \) Poles: 4

Why does this work?

- Row \( b/4 \) the zeros contains the even polynomial that is a factor of \( D(s) \)
- Row containing even polynomial to end of Routh table – test of ONLY the even polynomial

\[
\begin{array}{c|ccc}
\hline
s^5 & 1 & 6 & 8 \\
\hline
s^4 & 1 & 6 & 8 \\
\hline
s^3 & 4 & 12 & 0 \\
\hline
s^2 & 3 & 8 & 0 \\
\hline
s^1 & 1/3 & 0 & 0 \\
\hline
s^0 & 8 & 0 & 0 \\
\hline
\end{array}
\]

Example 6

\[
T(s) = \frac{s^3 + 7s^2 - 21s + 10}{s^6 + s^5 - 6s^4 + -s^2 - s + 6}
\]

| \( s^6 \) | 1 | -6 | -1 | 6 |
| \( s^5 \) | 1 | 0 | -1 | 0 |
| \( s^4 \) | 0 | 0 | 0 | 0 |
| \( s^3 \) | 0 | 0 | 0 | 0 |
| \( s^2 \) | 0 | 0 | 0 | 0 |
| \( s^1 \) | 0 | 0 | 0 | 0 |
| \( s^0 \) | 0 | 0 | 0 | 0 |
Stability Design via Routh – Hurwitz

Consider

\[
\frac{K}{s(s + 7)(s + 11)}
\]

Find the range of K for system to be stable assuming K > 0.

Step 1: Obtain transfer function

\[
T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}
\]

Step 2: Construct Routh Table

| s^3 | 1     | 77    |
| s^2 | 18    | K     |
| s^1 |       |       |
| s^0 |       |       |

For system to be stable: K < 77(18)

Stability in State Space

Given,

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

This system is stable if and only if \( \text{Re}(\lambda_i(A)) < 0 \)

for all \( i = 1, \ldots, n \)

Furthermore, if \( A \) satisfy the condition, we call \( A \) Hurwitz.