Root Locus Recap

- Root Locus: a method of presenting graphical information about a system’s behavior when the controller is working
  - Common tool for design of closed loop systems
  - Allows us to sketch out system behavior for a range of K

Example 1

Given the closed loop transfer function:

\[ \frac{K}{s^2 + 10s + K} \]

Look at poles for different values of K

<table>
<thead>
<tr>
<th>K</th>
<th>Pole 1</th>
<th>Pole 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10</td>
<td>0</td>
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<tr>
<td>5</td>
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<td>10</td>
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<td>45</td>
<td></td>
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<tr>
<td>50</td>
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</tbody>
</table>

Results for Example 1

Why the Root Locus Works?

- Closed loop transfer function
  \[ T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \]
- Poles occur when
  \[ KG(s)H(s) = -1 = 1 \angle (2k + 1)180^\circ \]
  - Must consider both magnitude and angle
Rules for the Root Locus

- Number of branches = close loop poles
  - Branch path that one pole traverses as the gain is varied
- Root Locus is symmetric about the real axis
- The root locus segments lie on the real axis to the left of an odd number of open loop poles and zeros
- The root locus begins (0 gain) at the poles and ends (∞ gain) at the zeros (finite and infinite) of \( G(s)H(s) \)
- Asymptotes
  \[
  \sigma_a = \sum_{\text{finite poles}} - \sum_{\text{finite zeros}} \frac{1}{2k + 1}
  \]
  \[
  \theta_a = \sum_{\text{finite poles}} - \sum_{\text{finite zeros}}
  \]
- Break-out & Break-in Points \( \frac{d[G(s)H(s)]}{ds} = 0 \)

Low Order Loci

- Use only the first few rules
  - Use the rules in order
- Practice sketching loci to gain proficiency
- The following are some examples of low order loci

One Pole

- \( 1/(s+2) \) w/ pole = -2

Two Poles

- \( 1/(s^2 + 6s + 8) \) with poles at -2, -4
One Zero, Two Poles

\[ \frac{s+3}{s^2 + 6s + 8} \] with one zero at -3 and poles at -2, -4

Zero Outside Two Poles

\[ \frac{s+5}{s^2 + 6s + 8} \] with one zero at -5 and poles at -2, -4

Three Poles

\[ \frac{1}{s^3 + 12s^2 + 44s + 48} \] with poles at -2, -4, -6

No Breakaway Points

\[ \frac{1}{s^3 + 8s^2 + 37s + 50} \] with poles at -2, -3 \( \pm j4 \)
With Breakaway Points

\[ \frac{1}{s^3 + 25s^2 + 193s + 169} \text{ with poles at } -1, -12 \pm j5 \]

One Zero, Three Poles

\[ \frac{s+2}{s^3 + 11s^2 + 34s + 24} \text{ with one zero at } -2 \text{ and poles at } -1, -4, -6 \]

A Few More Details on the Root Locus

- \( j\omega \)-axis crossings
  - Obtain this via Routh-Hurwitz criterion
  - Force a row of zeros in the Routh table to get this gain
  - Once we have the gain, we can solve for \( s \)

- Consider

\[ T(s) = \frac{K(s + 3)}{s^4 + 7s^3 + 14s^2 + (8 + K)s + 3K} \]

Transient Response Design via Gain Adjustment

- Design Procedure for Higher Order Systems
  1. Sketch the RL
  2. Assume 2nd order system w/ no zeros
  3. Find \( K \) to meet transient response specs
  4. Verify positions of higher order poles to make sure assumptions are valid
  5. If assumptions do not hold, simulate system numerically
Recall From Chapter 4

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
\[ s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d} \]
\[ T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \]
\[ \%OS = e^{\frac{\zeta\pi}{\sqrt{1 - \zeta^2}}} \times 100\% \]

Peak Time

- As \( \omega_d \) increases, \( T_p \) decreases

\[ T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d} \]

Settling Time

- As \( -\sigma_d \) increases, \( T_s \) decreases

\[ T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \]

Overshoot

- Let \( \theta \) be the angle made by \( -\sigma_d + j\omega_d \)
  - \( \sigma_d = \cos \theta \)
  - \( \%OS = e^{\zeta\pi/\sqrt{1 - \zeta^2}} \times 100\% \)
**In Summary**

**Conditions for Second-Order Approximations**
- Higher order poles are much farther to the left of the dominant 2nd order poles
- Closed-loop zeros near closed-loop 2nd order poles are canceled by proximity of higher-order closed-loop poles
- Closed-loop zeros not canceled by proximity of higher-order closed-loop poles are far from closed-loop 2nd order poles

**Example**
- Design value of $K$ such that we get 1.52% overshoot.
- Estimate settling time, peak time, and steady-state error.

**Generalized Root Locus**
- What about this system?
- Can we determine the behavior of the system for a range of $p_1$ values?
- Answer: Yes!
Root Locus for Positive Feedback Systems

1. Number of branches = 
   # of closed-loop poles

2. Symmetry

3. Real Axis Segments: to the left of even (vs. odd) # of finite poles and/or zeros

4. Starting & ending points

5. Asymptotes & intercepts
   \[ \sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{2N_a} \]
   \[ \theta_a = \frac{\text{finite poles} - \text{finite zeros}}{N_a} \]

6. Break-out & break-in points 
   \[ \frac{d}{ds}[G(s)H(s)] = 0 \]