**The Nyquist Criterion**

- Relates stability of the closed-loop system to the open-loop frequency response and open-loop pole location
- Gives # of closed-loop poles in RHP
- Similar to root-locus, we look at $1 + G(s)H(s) = 0$
- Unlike root-locus, we can simplify to $G(s)H(s)$

**The Concept of Mapping**

- Consider the following function: $F(s) = s - s_0$
  
  $s_0$ is possibly complex.

**Mapping Continued**

- Similarly, consider $F(s) = 1/(s - s_0)$ where $s_0$ is possibly complex.

**Mapping Continued**

- Suppose $F(s)$ is given by $F(s) = (s-s_0)(s-s_1)$

Note: # of zeros inside $C = $ # of times $\Gamma$ encircles the origin clockwise
Mapping Continued

- Next, suppose F(s) is given by
  \[ F(s) = \frac{1}{(s-s_0)(s-s_1)} \]

Note: # of zeros inside \( C \) = # of times \( \Gamma \) encircles the origin counterclockwise

Derivation of the Nyquist Criterion

Note the following:
1. Relationship between poles of \( 1+G(s)H(s) \) & poles of \( G(s)H(s) \)
2. Relationship between zeros of \( 1+G(s)H(s) \) & poles of \( T(s) \)

Let \( G(s) = \frac{N_G}{D_G} \) and \( H(s) = \frac{N_H}{D_H} \)

Note:
1) Poles of \( 1+GH \) = poles of \( GH \)
2) Zeros of \( 1+GH \) = poles of \( T \)

Derivation Continued

Let \( F(s) = 1+G(s)H(s) \)

The Nyquist Criterion

Given \( F(s) = 1 + G(s)H(s) \), recall

\[ G(s)H(s) = \frac{N_GN_H}{D_GD_H} \]
\[ 1 + G(s)H(s) = \frac{D_GD_H}{D_GD_H + N_GN_H} \]
\[ T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{D_GD_H}{N_GD_H} \]

Zeroes of \( F(s) \) correspond to the poles of \( T(s) \)!

Since \( N = P - Z \), if \( Z > 0 \) then system is unstable!
**Cauchy’s Principle of Argument**

Theorem:
Let \( F(s) \) be the ratio of 2 polynomials in \( s \). Let the closed curve \( C \) in the \( s \)-plane be mapped into the complex plane through the mapping \( F(s) \). If \( F(s) \) is analytic within and on \( C \), except at a finite number of poles, and if \( F(s) \) has neither poles nor zeros on \( C \), then

\[
N = P - Z
\]

where \( Z \) is the number of zeros of \( F(s) \) in \( C \), \( P \) is the number of poles in \( F(s) \) in \( C \), and \( N \) is the number of counterclockwise encirclements of the origin. (Phillips & Harbor)

**Some More Simplification**

Instead of \( F(s) = 1 + G(s)H(s) \), let \( F(s) = G(s)H(s) \)

Then, \( N = P - Z \) is the number of counterclockwise encirclements of \(-1\)

**Stability via Nyquist Criterion**

- \( Z = 0 \)
- \( P = 0 \)
- Thus, \( N = 0 \)

- \( Z = 2 \)
- \( P = 0 \)
- Thus, \( N = 2 \)

**A Closer Look**

- Counterclockwise \( \Rightarrow +1 \)
- Clockwise \( \Rightarrow -1 \)
Applying the Nyquist Criterion

- Use $Z = P - N$
- Consider

\[ \frac{R(s)}{E(s)} = \frac{K(s + 3)(s + 5)}{(s - 2)(s - 4)} C(s) \]

- Open-loop poles 2, 4
- Open-loop zeros -3, -5
- Question: For what range of $K$ is closed-loop system stable?

First note ...

Matlab: zpk, nyquist

Note the following:

- As $K$ changes, $\Gamma$ inflates/deflates
- For this system, $P = 2$, thus $N = P - Z$
  - If $Z = 0 \Rightarrow$ system is unstable
  - For system to be stable, $Z = 0 \Rightarrow N = 2$

To Find $K$

- Obtain Nyquist plot w/ $K = 1$
- Note, if Nyquist diagram intersections real-axis @ -1

\[ G(j\omega)H(j\omega) = -1 \]

System is marginally stable

- Let $-1/K$ be the critical point (instead of -1), find $K$ to satisfy Nyquist criteria

Example

Given

\[ G(s) = \frac{K}{s(s + 3)(s + 5)} \]

Step 1: Set $K = 1$, sketch poles & zeros in s-plane, plot Nyquist diagram

Step 2: Find $G(j\omega)H(j\omega)$

Step 3: Find point where Nyquist intersects negative real axis

Step 4: Determine $N$ for stability and then $K$
Stability and PhaseMargins

Gain Margin:
change in open-loop gain (dB) at 180° to make closed-loop system unstable

Phase Margin:
change in open-loop phase shift at unity gain to make closed-loop system unstable

Gain margin = \( G_M = 20 \log \alpha \)
Phase margin = \( \Phi_M = \alpha \)

Stability Range via Bode Plots
Given \( G(s) = \frac{K}{(s+5)(s+20)(s+50)} \), use Bode plots to find range of \( K \) where system is stable.

Evaluating Gain and Phase Margins

Close-loop Transient vs. Frequency Response

• Recall, 2nd order-system

\[
M_P = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}
\]

\[
\omega_P = \omega_n \sqrt{1 - 2\zeta^2}
\]
Phase Margin via Damping Ratio

- Consider the 2nd order open-loop TF
  \[ G(s) = \frac{\omega_n^2}{s(s + 2\zeta \omega_n)} \]

- Compute the Phase Margin, \( \Phi_M \):
  \[ \Phi_M = \tan^{-1} \left( \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \right) \]

Bandwidth of a System

- Frequency at which magnitude of response curve is 3dB below its value at 0 rad/s
  \[ \omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \]

Steady-State Error

- Unit Step Input
  \[ e(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)} \]

- Ramp Input
  \[ e(\infty) = \frac{1}{\lim_{s \to 0} sG(s)} \]

- Parabolic Input
  \[ e(\infty) = \frac{1}{\lim_{s \to 0} s^2G(s)} \]

- Position Constant
  \[ K_p = \lim_{s \to 0} G(s) \]

- Velocity Constant
  \[ K_v = \lim_{s \to 0} sG(s) \]

- Acceleration Constant
  \[ K_a = \lim_{s \to 0} s^2G(s) \]

Bode Plots & Steady-State Error Constants
Lead Compensation
- Passive analog of PD compensation
- Cascade compensator

\[ G_c(s) = \frac{1}{\beta} s + \frac{1}{\beta T} \]

The Effects of a Lead Compensator

Design Procedure for Lead Compensation
1. Find closed-loop bandwidth requirement to meet \( T_s, T_p \) or \( T_r \)
2. Set \( K \) s.t. uncompensated system satisfies steady-state error specs
3. Plot Bode plots for set \( K \) and determine uncompensated system’s phase margin
4. Find phase margin to meet \( \zeta \) or \%OS, find phase contribution from \( G_c \)
5. Determine \( \beta \)
6. Determine \( |G_c(j\omega)| \) @ peak of phase curve
7. Determine phase margin \( \phi \)
8. Design the break frequency for \( G_c \)
9. Reset system gain to compensate for \( G_c \)’s gain
10. Check bandwidth to ensure Step 1 specs are met
11. Simulate to check
12. Redesign if needed

Frequency Response of the Lead Compensator
Determine \( M_c(\omega), \phi_c(\omega) \), given

\[ G_c(s) = \frac{1}{\beta} s + \frac{1}{\beta T} \]

\[ M_c(\omega_{max}) = \sqrt{\frac{1}{\beta}} \]

\[ \phi_c(\omega_{max}) = \tan^{-1} \left( \frac{1 - \beta}{2\sqrt{\beta}} \right) = \sin^{-1} \left( \frac{1 - \beta}{1 + \beta} \right) \]
Example

Given \( G(s) = \frac{K}{s[(s+50)(s+120)]} \), design a lead compensator
s.t. \%OS = 20\%, \( T_S = 0.2s \), and \( K_v = 50 \).