Root Locus

- Root Locus: a method of presenting graphical information about a system’s behavior when the controller is working
  - Common tool for design of closed loop systems
  - Allows us to sketch out system behavior for a range of $K$
- Rules (negative feedback systems):
  - Number of branches = close loop poles
  - Root Locus is symmetric about the real axis
  - The root locus segments lie on the real axis to the left of an odd number of open loop poles and zeros
  - The root locus begins (0 gain) at the poles and ends (∞ gain) at the zeros (finite and infinite) of $G(s)H(s)$
- Asymptotes
  - $\alpha = \frac{\sum$finite poles - infinite zeros $}{\sum$finite zeros $(2k+1)\pi}$
  - $\theta = \frac{\sum$finite poles - infinite zeros $}{\sum$finite zeros $\pi}$
- Break-out & Break-in Points
  - $d \frac{[G(s)H(s)]}{ds} = 0$

Translant Response Design via Gain Adjustment

- Design Procedure for Higher Order Systems
  1. Sketch the RL
  2. Assume 2nd order system w/ no zeros
  3. Find $K$ to meet transient response specs
  4. Verify positions of higher order poles to make sure assumptions are valid
  5. If assumptions do not hold, simulate system numerically

Compensator Configurations

- Cascade
  - Cascade compensator
  - Original controller
  - Plant

- Feedback
  - Original controller
  - Plant
  - Feedback compensator

PID Controller Design

- $G_c(s) = K_1 + \frac{K_2}{s} + K_3s$
- 1. Evaluate uncomp sys to get determine desired transient
- 2. Design PD controller
- 3. Simulate to check
- 4. Redesign if necessary
- 5. Design PI controller to yield desired steady-state error
- 6. Determine $K_1$, $K_2$, and $K_3$
- 7. Simulate to check
- 8. Redesign if necessary
**PID Controller Design**

\[ G_c(s) = K_1 + \frac{K_2}{s} + K_3s \]

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**Feedback Compensation**

- Disadvantage: More complicated
- Advantage: Faster response
  - Sometimes physical system characteristics does not allow us to use cascade compensators
  - Often does not require additional amplification
  - Two Approaches

**Approach 1**

- Design a minor loop’s transient response separately from the closed-loop system response.
- Example 3:
  - \( \zeta = 0.8 \) for minor
  - \( \zeta = 0.6 \) for closed-loop

**Approach 2**

- Design a minor loop’s transient response separately from the closed-loop system response.
**Frequency Response**

**Definition:**
- The *frequency response* of a linear system is the relationship between the gain and the phase of a sinusoidal input and the corresponding sinusoidal output.

Note:
- Frequency response gives steady state response
- Complements analysis from root locus
- Often used to back out system parameters
- Not so good for transient analysis

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**From Last Time**

- Frequency Response
- Leads to 2 plots
  - $M_G$ vs. $\omega$
  - $20\log_{10}(M_G)$ vs. $\log_{10}(\omega)$
  - $\phi_G$ vs. $\omega$
  - $20\log_{10}(M_G)$ vs. $\log_{10}(\omega)$

**Bode Plots**

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**Nyquist Criterion**

Instead of $F(s) = 1 + G(s)H(s)$, let $F(s) = G(s)H(s)$

Then, $N = P - Z$ is the number of counterclockwise encirclements of -1

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**Using Nyquist Criterion to Find K**

Given: $G(s) = \frac{K}{s(s + 3)(s + 5)}$

Step 1: Set $K = 1$, sketch poles & zeros in s-plane, sketch Nyquist diagram

Step 2: Find $G(j\omega)H(j\omega)$

Step 3: Find point where Nyquist intersects negative real axis

Step 4: Determine N for stability and then K
### Gain and Phase Margins

**Gain Margin:**
- change in open-loop gain (dB) at 180° to make closed-loop system unstable

**Phase Margin:**
- change in open-loop phase shift at unity gain to make closed-loop system unstable

### Frequency Response of the Lead Compensator

Determine $M_c(\omega)$, given

### Design Procedure for Lead Compensation

1. Find closed-loop bandwidth requirement to meet $T_s$, $T_p$ or $T_r$
2. Set $K$ s.t. uncompensated system satisfies steady-state error specs
3. Plot Bode plots for set $K$ and determine uncompensated system's phase margin
4. Find phase margin to meet $\zeta$ or %OS, find phase contribution from $G_c$
5. Determine $\beta$
6. Determine $|G_c(j\omega)|$ @ peak of phase curve
7. Determine phase margin $\phi$
8. Design the break frequency for $G_c$
9. Reset system gain to compensate for $G_c$'s gain
10. Check bandwidth to ensure Step 1 specs are met
11. Simulate to check
12. Redesign if needed

### Design Procedure for Lag Compensator

- $G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$

1. Set $K$ to satisfy steady-state specification & plot Bode diagrams for selected $K$
2. Find $\omega_d$ where such that $\Phi_M$ is $5^\circ$-$12^\circ$ greater than $\Phi_M(\omega_d)$
3. Set $|G_c(j\omega_d)|$ s.t Bode plot for $G_c(j\omega)G(j\omega)$ goes through 0dB at $\omega_d$
4. Set upper break freq. @ 1 decade below $\omega_d$
5. Low freq. asymptote to be at 0 dB
6. Connect low + high freq. asymptote via -20 dB/decade line to locate low break freq.
7. Reset $K$ to compensate for any attenuation from $G_c$ to maintain steady-state specs
**PI Compensation w/ Bode Plots**

\[ G_c(s) = \frac{K_1 + \frac{K_2}{s}}{s} = \frac{K_1(s + \frac{K_2}{s})}{s} \]

1. Set \( K \) to meet steady-state spec
2. Determine the phase contribution of \( G_c \) and thus \( \Phi_{M,\text{comp}} \)
3. Plot Bode plots for \( G(s) \) for \( K \) chosen in Step 1
4. Find \( \omega \) and magnitude (dB) s.t. phase angle is \((-180^\circ + \Phi_{M,\text{comp}})\)
5. Set break freq. to be 0.1 \( \omega \)
6. Set \( K_1 \) s.t. magnitude of response is 0 dB at \( \omega \)

**PD Compensation w/ Bode Plots**

\[ G_c(s) = K_1 + \frac{K_2}{s} = K_2(\frac{s + K_1}{K_2}) \]

1. Find closed-loop \( \omega_{WB} \) to meet \( T_p, T_r, \) or \( T_s \)
2. Set \( K \) to meet steady-state
3. Pick the \( \omega_{WB,\text{new}} = \omega_{WB} + \omega_{\text{correction}} \) where \( \omega_{\text{correction}} \) is set by the designer
4. Find the phase angle at the new \( \omega_{WB,\text{new}} \) (given in 3)
5. Find the contribution of the compensator = \(-180^\circ + \Phi_{\omega_{WB,\text{new}}} + \Phi_{M}(\zeta_d)\)
6. Determine \( K_1/K_2 \) based on the angle found in 5
7. Set \( K_2 \) such that DC gain of compensator is unity

**Z-Transforms**

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(z) )</th>
<th>( f(KT) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(t) )</td>
<td>( \frac{z}{z-1} )</td>
<td>( u(KT) )</td>
</tr>
<tr>
<td>( t )</td>
<td>( \frac{z}{z^2} )</td>
<td>( KT )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{z}{z-e^{-aT}} )</td>
<td>( e^{-at} )</td>
</tr>
<tr>
<td>( e^{-at} ) ( e^{+z} )</td>
<td>( z ) ( z )</td>
<td>( (z)^t e^{-at} )</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{z^2 + \omega^2} )</td>
<td>( \sin \omega KT )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>( \frac{z + \omega}{z^2 - 2\omega \cos \omega T} )</td>
<td>( \cos \omega KT )</td>
</tr>
<tr>
<td>( e^{-at} \sin \omega t )</td>
<td>( \frac{\omega}{z^2 + \omega^2} )</td>
<td>( e^{-at} \sin \omega KT )</td>
</tr>
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<td>( \frac{z + \omega}{z^2 - 2\omega \cos \omega T} )</td>
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</tbody>
</table>

**Z-Transform Theorems**

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Name</th>
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<tbody>
<tr>
<td>( z(tf(t)) = zF(z) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>( z(f(t) + f(t)) = F(z) + F(z) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>( z(e^{-at} f(t)) = F(e^{-aT})z )</td>
<td>Complex differentiation</td>
</tr>
<tr>
<td>( z(f(i - nT)) = z^nf(z) )</td>
<td>Real translation</td>
</tr>
<tr>
<td>( z(f(t)) = -\frac{1}{T} \frac{dF(z)}{dz} )</td>
<td>Complex differentiation</td>
</tr>
<tr>
<td>( f(0) = \lim_{z \to 0} z^{-n} f(z) )</td>
<td>Initial value theorem</td>
</tr>
<tr>
<td>( f(\infty) = \lim_{z \to 1} (1 - z)^{-1} F(z) )</td>
<td>Final value theorem</td>
</tr>
</tbody>
</table>

Note: \( iT \) may be substituted for \( z \) in the table.
Stability

- Stability via the s-Plane
  - Routh-Hurwitz criterion for stability
  - More helpful to have transformations that are linear
    - Bilinear Transformations between s-Plane and z-Plane
    \[
    s = \frac{z + 1}{z - 1} \\
    z = \frac{s + 1}{s - 1}
    \]

Transient Performance Specifications

In the s-plane
- \( T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d} \)
- \( T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d} \)
- \( %OS = e^{-\zeta \pi / \sqrt{1 - \zeta^2}} \times 100\% \)
Cascade Compensation
- Design via s-plane
- Transform controller into z-plane
  - Transformation that preserves behavior of continuous compensator
- Tustin Transformations:
  - Bilinear transformation that yields digital transfer function whose output matches analog version at the sampling instants

\[
\begin{align*}
 s &= \frac{2z - 1}{Tz + 1} \\
 z &= \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}
\end{align*}
\]

Choosing T
- If T is too large (or too low sampling frequency)
- In general, upper bound on T should be

\[
\frac{0.15}{\omega_M} \leq T \leq \frac{0.5}{\omega_M}
\]

Pole Placement
- Given \(G(s) = \frac{20(s+5)}{s(s+1)(s+4)}\), design phase-variable feedback gains to yield 9.5% overshoot with \(T_s = 0.74\) sec
- Locations of poles: \(-5.4 \pm j7.2, -5.1\)