Problem 6-9. [1]

Derive the equations of motion for the simple pendulum whose support point $O'$ moves in a circular path of radius $r$ about a fixed center $O$. The bob has a mass $m$. All other rigid bodies are massless. Use $\theta$ and $\phi$ as generalized coordinates and their derivatives as generalized speeds. See figure.

Let $u_1 = \dot{\phi}$ and $u_2 = \dot{\theta}$, then

$$x = r \sin \phi + l \sin \theta, \quad \dot{x} = \dot{\phi} r \cos \phi + \dot{\theta} l \cos \theta = u_1 r \cos \phi + u_2 l \cos \theta,$$

$$y = r \cos \phi + l \cos \theta, \quad \dot{y} = -\dot{\phi} r \sin \phi - \dot{\theta} l \sin \theta = -u_1 r \sin \theta - u_2 l \sin \theta.$$

Then, the kinetic energy of the system is given by

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m \{(u_1 r \cos \phi + u_2 l \cos \theta)^2 + (-u_1 r \sin \theta - u_2 l \sin \theta)^2\} = \frac{1}{2} m \left(u_1^2 r^2 + u_2^2 l^2 + 2rlu_1u_2 \cos(\phi - \theta)\right).$$

with the potential energy given by

$$V = -mgy = -mg(r \cos \phi + l \cos \theta).$$

Then, the Lagrangian, denoted by $L$, is

$$L = T - V$$

$$= \frac{1}{2} m \left(u_1^2 r^2 + u_2^2 l^2 + 2rlu_1u_2 \cos(\phi - \theta)\right) + mg(r \cos \phi + l \cos \theta)$$

The equations of motion for the system is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$
for $i = 1, 2$ and $q_i$ denotes the $i^{th}$ generalized coordinate. Thus,

$$
\frac{\partial L}{\partial u_1} = \frac{1}{2} m \left( 2r^2 u_1 + 2r l u_2 \cos(\phi - \theta) \right), \\
\frac{\partial L}{\partial u_2} = \frac{1}{2} m \left( 2l^2 u_2 + 2rl u_1 \cos(\phi - \theta) \right), \\
\frac{\partial L}{\partial \phi} = \frac{1}{2} m \left( -2r l u_1 u_2 \sin(\phi - \theta) \right) - mgr \sin \phi, \\
\frac{\partial L}{\partial \theta} = \frac{1}{2} m \left( 2r l u_1 u_2 \sin(\phi - \theta) \right) - mgl \sin \theta, \\
\frac{d}{dt} \frac{\partial L}{\partial u_1} = \frac{1}{2} m \left( 2r^2 \ddot{u}_1 + 2r l \ddot{u}_2 \cos(\phi - \theta) - 2r l u_2 \sin(\phi - \theta) \right) \left( u_1 - u_2 \right), \\
\frac{d}{dt} \frac{\partial L}{\partial u_2} = \frac{1}{2} m \left( 2l^2 \ddot{u}_2 + 2r l \ddot{u}_1 \cos(\phi - \theta) - 2r l u_1 \sin(\phi - \theta) \right) \left( u_1 - u_2 \right).
$$

Combining the above equations gives

$$
\frac{d}{dt} \frac{\partial L}{\partial u_1} - \frac{\partial L}{\partial \phi} = 0, \\
r^2 \ddot{u}_1 + r l \ddot{u}_2 \cos(\phi - \theta) + r l u_2^2 \sin(\phi - \theta) + gr \sin \phi = 0.
$$

This is the first equation of motion, with the second equation of motion given by

$$
\frac{d}{dt} \frac{\partial L}{\partial u_2} - \frac{\partial L}{\partial \theta} = 0, \\
l^2 \ddot{u}_2 + r l \dot{u}_1 \cos(\phi - \theta) - r l u_1^2 \sin(\phi - \theta) + gl \sin \theta = 0.
$$

References