Second – Order Systems

- Given, \( G(s) = \frac{1}{s^2 + bs + c} \) and \( R(s) = \frac{1}{s} \), we know
  \[
  C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{s + r_2}
  \]

- Possible solutions
  - \( r_1 \) and \( r_2 \) are real & distinct
  - \( r_1 \) and \( r_2 \) are real & repeated
  - \( r_1 \) and \( r_2 \) are both imaginary
  - \( r_1 \) and \( r_2 \) are complex conjugates

Case 1: Real & Distinct Roots

\[
C(s) = \frac{1}{s(s^2 + bs + c)} = \frac{K_1}{s} + \frac{K_2}{s + r_1} + \frac{K_3}{s + r_2}
\]

This gives

\[
c(t) = K_1 e^{-r_1 t} + K_2 e^{-r_2 t} + K_3 e^{-r_1 t}
\]

Overdamped response
Case 2: Real & Repeated Roots

\[ c(t) = K_1 + K_2 e^{-\xi t} + K_3 e^{-\xi t} \]

This gives a critically damped response.

Case 3: All Imaginary Roots

\[ c(t) = K_1 + K_4 \cos(\omega t - \phi) \]

Gives an undamped response.

Case 4: Roots Are Complex

\[ c(t) = K_1 + K_2 e^{-\omega_0 t} \cos(\omega t - \phi) \]

Gives an underdamped response.

General Underdamped Systems

Given \( G(s) = \frac{u_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} \) and \( R(s) = \frac{1}{s} \) such that \( \zeta < 0 \) and \( \omega_0 > 0 \), using PFE we obtain

\[ c(t) = K_1 \left( \frac{\sigma + \zeta \omega_0}{\sqrt{\zeta^2 + \omega_0^2}} \sqrt{1 - \zeta^2} \right) \frac{1}{\sigma^2 + 2\zeta \omega_0 \sigma + \omega_0^2} + K_2 \frac{\omega_0^2}{\sigma^2 + 2\zeta \omega_0 \sigma + \omega_0^2} \]

Taking the inverse Laplace results in

\[ c(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\omega_0 t} \cos\left(\omega_0 \sqrt{1 - \zeta^2} t - \phi\right) \]

with \( \phi = \tan^{-1} \left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \).
Characterizing Underdamped Systems

Peak Time: \( T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \)

Percent Overshoot: \( \%OS = e^{-\pi \sqrt{1 - \zeta^2}} \times 100\% \)

(Note: given %OS, we can use the equation to solve for \( \zeta \))

Settling Time: \( T_s = \frac{4}{\zeta \omega_n} \)

Rise Time: obtained numerically

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Characterizing Underdamped Systems

As \( \omega_d \) increases, \( T_p \) decreases

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Peak Time

As \( \omega_d \) increases, \( T_p \) decreases

\( T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_d} \)
**Settling Time**

As $-\sigma_d$ increase, $T_s$ decreases.

\[ T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d} \]

**Overshoot**

- Let $\theta$ be the angle made by $-\sigma_d + j\omega_d$
- $\sigma_d = \cos \theta$

\[ \%OS = e^{-\zeta \sqrt{1 - \zeta^2}} \times 100\% \]

**In Summary**

Higher Order Systems

Approximation via 2nd order system?
- Yes, only under certain conditions
Higher Order Systems

Consider

\[ C(s) = \frac{A}{s^4} + \frac{B(s + C_0)}{(s + \alpha_1)(s + \alpha_2)} + \frac{C}{s + \alpha_3} + \frac{D}{s + \alpha_4} \]

Dominant Poles

\[ \alpha_F \gg \zeta \omega_N \]

Stability

- Total response of the system is
  \[ e(t) = e_{\text{forced}}(t) + e_{\text{natural}}(t) \]

- A linear time-invariant (LTI) system is
  - Stable if \( e_{\text{natural}}(t) \to 0 \) as \( t \to \infty \)
  - Unstable if \( e_{\text{natural}}(t) \to \infty \) as \( t \to \infty \)
  - Marginally stable if \( e_{\text{natural}}(t) \) neither grows or decays as \( t \to \infty \)

- Another definition – Bounded Input Bounded Output (BIBO)
  - A system is stable if every bounded input yields a bounded output
  - A system is unstable if every bounded input yields an unbounded output

\[ \delta - \varepsilon \] relationship of stability

- Stability in the sense of Lyapunov
  - \( C(t) \) is stable if and only if for any \( \varepsilon > 0 \), \exists \delta > 0 \) such that
    \[ \|e(0)\| < \delta \]
    \[ \|e(t)\| < \varepsilon, \quad t > 0 \]

- Asymptotic Stability
  - \( C(t) \) is asymptotically stable if and only if for any \( \delta > 0 \) such that
    \[ \|e(0)\| < \delta \]
    \[ \|e(t)\| \to 0, \quad \text{as} \ t \to \infty \]

Stability & Location of Poles
### Routh-Hurwitz
- Provides stability information without requiring to explicitly solve for poles
- Trivial for analysis
- Provides bounds for design
- 2 Steps:
  - Step 1: Generate Routh table
  - Step 2: Interpret the Routh table following Routh-Hurwitz criterion

### Generating a Basic Routh Table

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>$a_4$</th>
<th>$a_3$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>$a_3$</td>
<td>$a_2$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$s^2$</td>
<td>$a_2$ $a_1$</td>
<td>$b_1$</td>
<td>$a_1$ $a_0$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$b_1$ $b_0$</td>
<td>$c_1$</td>
<td>$b_0$ $b_0$</td>
</tr>
<tr>
<td>$s^0$</td>
<td>$c_1$</td>
<td>$d_1$</td>
<td>$c_0$ $c_0$</td>
</tr>
</tbody>
</table>

### Routh-Hurwitz Criterion
- # of roots located in the RHP == # of sign changes in the 1st column of Routh table

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
<th>31</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>10</td>
<td>1030</td>
<td>0</td>
</tr>
<tr>
<td>$s^2$</td>
<td>-72</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s^1$</td>
<td>103</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Example 4

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$
Another Method to Handle 0s in 1st Column

- Fact:
  - $R_1 = \text{Roots(Polynomial 1)}$
  - $R_2 = \text{Roots(Polynomial 2)}$
  - For every $r_i \in R_1$ and $r_j \in R_2$, $r_i = 1/r_j$ for all $i = 1, \ldots, p$
  - Then # of $r_i$ in RHP & LHP == # of $r_j$ in RHP & LHP

- Given: $a^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$
- Let $s = 1/d$, then
  \[
  \left(\frac{1}{d}\right)^n \left[1 + a_{n-1}d + \cdots + a_1d^{n-1} + a_0d^n\right] = 0
  \]

Entire Row in Routh Table is 0

- From our previous example, the row immediately before the row of 0s is
  \[
  P(s) = s^4 + 6s^2 + 8
  \]
  then, compute $dP/ds = 4s^3 + 12s + 0$
- Replace the row of zeros with the coefficients of $dP/ds$

Example 4 – Again

- $T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$
- $D(d) = 3d^5 + 5d^4 + 6d^3 + 3d^2 + 2d + 1$

| $d^5$ | 3 | 6 | 2 |
| $d^4$ | 5 | 3 | 1 |
| $d^3$ |
| $d^2$ |
| $d^1$ |
| $d^0$ |

Why does this work?

- A row of zeros appear when a purely even or purely odd polynomial is a factor of $D(s)$
  - Ex: $s^4 + 5s^2 + 7$ – even
  - $s^5 + 5s^3 + 7s + 1$ – other
- Even polynomials only have roots that are symmetrical about the origin
Why does this work?

- Row b/4 the zeros contains the even polynomial that is a factor of \( D(s) \)
- Rows containing even polynomial to end of Routh table – test of ONLY the even polynomial

| \( s^5 \) | 1 | 6 | 8 |
| \( s^4 \) | 1 | 6 | 8 |
| \( s^3 \) | 4 | 8 | 0 |
| \( s^2 \) | 3 | 8 | 0 |
| \( s^1 \) | 1/3 | 0 | 0 |
| \( s^0 \) | 8 | 0 | 0 |

Furthermore

\[ T(s) = s^6 + s^5 + 12s^4 + 22s^3 + 39s^2 + 59s + 48s^2 + 38s + 20 \]

| \( s^6 \) | 1 | 12 | 39 | 48 | 20 |
| \( s^5 \) | 1 | 22 | 59 | 38 | 0 |
| \( s^4 \) | -1 | -2 | 1 | 2 | 0 |
| \( s^3 \) | 1 | -3 | 2 | 0 | 0 |
| \( s^2 \) | 8 | 2 | 0 | 0 | 0 |
| \( s^1 \) | 3 | 4 | 0 | 0 | 0 |
| \( s^0 \) | 4 | 0 | 0 | 0 | 0 |

- Odd RHP Poles: 2
- Odd LHP Poles: 2
- Even RHP Poles: 0
- Even LHP Poles: 0
- Even \( j\omega \) Poles: 4

Stability Design via Routh – Hurwitz

- Consider \( \frac{R(s)}{C(s)} = \frac{K}{s^3 + 18s^2 + 77s + K} \)

- Find the range of \( K \) for system to be stable assuming \( K > 0 \).

- Step 1: Obtain transfer function \[ T(s) = \frac{K}{s^3 + 18s^2 + 77s + K} \]

- Step 2: Construct Routh Table

- Want \( K > 0 \) s.t. system is stable

| \( s^3 \) | 1 | 77 |
| \( s^2 \) | 18 | \( K \) |
| \( s^1 \) | \( s^0 \) |

For system to be stable: \( K < 77(18) \)
Three Specifications for Analysis & Design

- Transient Response (Chapter 4)
  - Performance specifications for 1st and 2nd order systems

- Stability (Chapter 6)
  - Pole locations
  - Routh-Hurwitz Method

- Steady-State Errors (Chapter 7)