Localization

Example

- Problem
  - Estimating a robot's coordinates relative to an external reference frame.

- Given:
  - Environment w/ 3 doors
  - Robot is given a map of this environment

- Task:
  - Find out where on this map it is through sensing & motion.

Robot Localization Example

\( p(x) \)

\( \text{bel}(x) \)
State Estimation

- Core Idea: Estimate *state* from sensor data.
- Estimation of quantities from sensor data that are *not directly* observable, but can be *inferred*.
- Building *belief distributions* over possible world states.
- Probabilistic inference:
  - Process of calculating possible values of random variables that are derived from other random variables & observed data.

Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$
- If $X$ and $Y$ are independent then
  
  $P(x,y) = P(x) P(y)$

- $P(x \mid y)$ is the probability of $x$ given $y$
  
  $P(x \mid y) = \frac{P(x,y)}{P(y)}$
  
  $P(x,y) = P(x \mid y) P(y)$

- If $X$ and $Y$ are independent then
  
  $P(x \mid y) = P(x)$

Law of Total Probability, Marginals

<table>
<thead>
<tr>
<th>Discrete case</th>
<th>Continuous case</th>
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</thead>
<tbody>
<tr>
<td>$\sum_x P(x) = 1$</td>
<td>$\int p(x) , dx = 1$</td>
</tr>
<tr>
<td>$P(x) = \sum_y P(x,y)$</td>
<td>$p(x) = \int p(x,y) , dy$</td>
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<tr>
<td>$P(x) = \sum_y P(x \mid y) P(y)$</td>
<td>$p(x) = \int p(x \mid y) p(y) , dy$</td>
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Bayes Formula

\[
P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)}
\]

\[
P(x,y) = P(x \mid y) P(y) = P(y \mid x) P(x)
\]

\[
\Rightarrow
\]

\[
P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}
\]
Normalization

\[ P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} \]
\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)} \]

Algorithm:

\[ \forall x: \text{aux}_{xy} = P(y \mid x) P(x) \]
\[ \eta = \sum_x \text{aux}_{xy} \]
\[ \forall x: P(x \mid y) = \eta \text{aux}_{xy} \]

Conditioning

- Law of total probability:

\[ P(x) = \int P(x, z) dz \]
\[ P(x) = \int P(x \mid z) P(z) dz \]
\[ P(x \mid y) = \int P(x \mid y, z) P(z \mid y) dz \]

Bayes Rule with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z) P(x \mid z)}{P(y \mid z)} \]

Conditional Independence

\[ P(x, y \mid z) = P(x \mid z) P(y \mid z) \]

Equivalent to

\[ P(x \mid z) = P(x \mid z, y) \]
\[ P(y \mid z) = P(y \mid z, x) \]
Assumptions on State

• *x* is complete if it is the best predictor of the future
  ➢ Knowledge of past states, measurements, or controls DOES NOT improve accuracy of prediction
  ➢ Says nothing about whether future is deterministic or stochastic

• If future is stochastic, then processes that meet these conditions are called Markov chains
• In theory – we assume complete states
• In practice – no state is really complete

Probabilistic Generative Laws

• Evolution of state and measurements is governed by probabilistic laws

• Measurements / Observations/ Precepts
  ➢ *z*₁, *z*₂ = *z*ᵣ, *z*ᵣ₊₁, *z*ᵣ₊₂, ⋯, *z*ᵣ₂

• Measurement Probability

\[
P(z_t \mid x_t)
\]

Simple Example of State Estimation

• Suppose a robot obtains measurement *z*
• What is *P(open|z)*?

Causal vs. Diagnostic Reasoning

• *P(open|z)* is diagnostic.
• *P(z|open)* is causal.
• Often causal knowledge is easier to obtain.
• Bayes rule allows us to use causal knowledge:

\[
P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}
\]
Example

- \( P(z|\text{open}) = 0.6 \quad P(z|\neg\text{open}) = 0.4 \)
- \( P(\text{open}) = P(\neg\text{open}) = 0.5 \)
- What is \( P(\text{open}|z) \)?

\[
P(\text{open} | z) = \frac{P(z | \text{open}) P(\text{open})}{P(z | \text{open}) P(\text{open}) + P(z | \neg\text{open}) P(\neg\text{open})}
\]

\[
P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67
\]

- \( z \) raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation \( z_2 \).
- How can we integrate this new information?
- More generally, how can we estimate \( P(x | z_1 \ldots z_n) \)?

Recursive Bayesian Updating

\[
P(x | z_1, \ldots, z_n) = \frac{P(z_1 | x, z_2, \ldots, z_n) P(x | z_1, \ldots, z_{n-1})}{P(z_1 | z_2, \ldots, z_n)}
\]

**Markov assumption**: \( z_1 \) is independent of \( z_2, \ldots, z_n \) if we know \( x \).

\[
P(x | z_1, \ldots, z_n) = \frac{P(z_1 | x) P(x | z_1, \ldots, z_{n-1})}{P(z_1 | z_2, \ldots, z_n)}
\]

\[
= \eta P(z_1 | x) P(x | z_1, \ldots, z_{n-1})
\]

\[
= \eta \prod_{i=1}^{n} P(z_i | x) P(x)
\]

Example: Second Measurement

- \( P(z_2|\text{open}) = 0.5 \quad P(z_2|\neg\text{open}) = 0.6 \)
- \( P(\text{open}|z_2) = 2/3 \)

\[
P(\text{open} | z_1, z_2) = \frac{P(z_2 | \text{open}) P(\text{open} | z_1)}{P(z_2 | \text{open}) P(\text{open} | z_1) + P(z_2 | \neg\text{open}) P(\neg\text{open} | z_1)}
\]

\[
= \frac{\frac{2}{5} \cdot \frac{1}{3}}{\frac{2}{5} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{2}{3}} = \frac{5}{8} = 0.625
\]

- \( z_2 \) lowers the probability that the door is open.
A Typical Pitfall

- Two possible locations $x_1$ and $x_2$
- $P(x_1) = 0.99$
- $P(z|x_2) = 0.09 \quad P(z|x_1) = 0.07$

Actions

- Often the world is dynamic since
  - actions carried out by the robot.
  - actions carried out by other agents.
  - or just the time passing by
  - ... change the world.

- How can we incorporate such actions?

Typical Actions

- The robot turns its wheels to move
- The robot uses its manipulator to grasp an object
- Plants grow over time...

- Actions are never carried out with absolute certainty.
- In contrast to measurements, actions generally increase the uncertainty.

Modeling Actions

- To incorporate the outcome of an action $u$ into the current “belief”, we use the conditional pdf

\[ P(x|u,x') \]

- This term specifies the pdf that executing $u$ changes the state from $x'$ to $x$. 
Example: Closing the door

State Transitions

- $P(x|u,x')$ for $u = "close\ door"$:

- If the door is open, the action “close door” succeeds in 90% of all cases.

- State Transition Probability: $P(x_t | x_{t-1}, u_t)$