Normalization

\[ P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \eta P(y \mid x) P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)} \]

Algorithm:

\[ \forall x: \text{aux}_{xy} = P(y \mid x) P(x) \]

\[ \eta = \frac{1}{\sum_x \text{aux}_{xy}} \]

\[ \forall x: P(x \mid y) = \eta \text{aux}_{xy} \]

Causal vs. Diagnostic Reasoning

- \( P(\text{open} \mid z) \) is diagnostic.
- \( P(z \mid \text{open}) \) is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

\[ P(\text{open} \mid z) = \frac{P(z \mid \text{open}) P(\text{open})}{P(z)} \]

Simple Example of State Estimation

- Suppose a robot obtains measurement \( z \)
- What is \( P(\text{open} \mid z) \)?

Example

- \( P(z \mid \text{open}) = 0.6 \)
- \( P(z \mid \neg \text{open}) = 0.3 \)
- \( P(\text{open}) = P(\neg \text{open}) = 0.5 \)
- What is \( P(\text{open} \mid z) \)?

\[ P(\text{open} \mid z) = \frac{P(z \mid \text{open}) P(\text{open})}{P(z \mid \text{open}) P(\text{open}) + P(z \mid \neg \text{open}) P(\neg \text{open})} \]

\[ P(\text{open} \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67 \]

\( z \) raises the probability that the door is open
Combining Evidence

• Suppose our robot obtains another observation $z_2$.
• How can we integrate this new information?
• More generally, how can we estimate $P(x \mid z_1...z_n)$?

Recursive Bayesian Updating

\[
P(x \mid z_1,...,z_n) = \frac{P(z_n \mid x, z_1,...,z_{n-1}) P(x \mid z_1,...,z_{n-1})}{P(z_n \mid z_1,...,z_{n-1})}
\]

Markov assumption: $z_n$ is independent of $z_1,...,z_{n-1}$ if we know $x$.

\[
P(x \mid z_1,...,z_n) = \eta P(z_n \mid x) P(x \mid z_1,...,z_{n-1})
\]

\[
= \eta \prod_{i=1}^{n} P(z_i \mid x) P(x)
\]

Actions

• Often the world is dynamic since
  > actions carried out by the robot.
  > actions carried out by other agents.
  > or just the time passing by
  ... change the world.

• How can we incorporate such actions?

Example

\[
bel(X_0=open) = bel(X_0 = closed) = 0.5
\]
\[
p(Z_t=sense_open \mid X_t=is_open) = 0.6
\]
\[
p(Z_t=sense_closed \mid X_t=is_open) = 0.4
\]
\[
p(Z_t=sense_open \mid X_t=is_closed) = 0.2
\]
\[
p(Z_t=sense_closed \mid X_t=is_closed) = 0.8
\]
\[
p(X_t=is_open \mid U_t=push, X_{t-1}=is_open) = 1
\]
\[
p(X_t=is_closed \mid U_t=push, X_{t-1}=is_open) = 0
\]
\[
p(X_t=is_open \mid U_t=do_nothing, X_{t-1}=is_open) = 1
\]
\[
p(X_t=is_closed \mid U_t=do_nothing, X_{t-1}=is_open) = 0
\]
\[
p(X_t=is_closed \mid U_t=push, X_{t-1}=is_closed) = 0.2
\]
\[
p(X_t=is_open \mid U_t=do_nothing, X_{t-1}=is_closed) = 1
\]
\[
p(X_t=is_closed \mid U_t=do_nothing, X_{t-1}=is_closed) = 0
\]

$t=1$: $U_1 = do_nothing$, $Z_1 = sense_open$

$t=2$: $U_2 = push$, $Z_2 = sense_open$
Bayes Filters: Framework

- **Given:**
  - Stream of observations \( z \) and action data \( u \):
    \[ d_i = \{u_i, z_1, \ldots, u_i, z_i\} \]
  - Sensor model \( P(z|x) \).
  - Action model \( P(x|u,x') \).
  - Prior probability of the system state \( P(x) \).

- **Wanted:**
  - Estimate of the state \( X \) of a dynamical system.
  - The posterior of the state is also called **Belief**:
    \[
    Bel(x_i) = P(x_i | u_i, z_1, \ldots, u_i, z_i)
    \]

Markov Assumption

- **Underlying Assumptions**
  - Static world
  - Independent noise
  - Perfect model, no approximation errors

\[
\begin{align*}
    p(z_i | x_{i-1}, z_{i-1}, u_{i-1}) &= p(z_i | x_i) \\
    p(x_i | x_{i-1}, z_{i-1}, u_{i-1}) &= p(x_i | x_{i-1}, u_i)
\end{align*}
\]

Bayes Filters are Familiar!

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

\[
\begin{align*}
    Bel(x_i) &= \eta \cdot P(z_i | x_i) \int P(x_i | u_i, x_{i-1}) \, dx_{i-1}
\end{align*}
\]

**Algorithm** Bayes_filter( Bel(x), d ):

1. \( \eta \leftarrow 0 \)
2. If \( d \) is a perceptual data item \( z \) then
   3. For all \( x \) do
   4. \( Bel'(x) = P(z_i | x_i) Bel(x_i) \)
   5. \( \eta = \eta + Bel'(x) \)
   7. For all \( x \) do
   8. \( Bel'(x) = \eta^{-1} Bel'(x) \)
9. Else if \( d \) is an action data item \( u \) then
   10. For all \( x \) do
   11. \( Bel'(x) = \int P(x_i | u_i, x_{i-1}) \, dx' \)
12. Return \( Bel'(x) \)
Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Gaussians

Univariate

\[ p(x) \sim N(\mu, \sigma^2) : \]
\[
 p(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right)
\]

Multivariate

\[ p(x) \sim N(\mu, \Sigma) : \]
\[
 p(x) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right)
\]

Properties of Gaussians

\[ X \sim N(\mu, \sigma^2) \]
\[ Y = aX + b \]
\[ \Rightarrow Y \sim N(a\mu + b, a^2\sigma^2) \]

\[ X_1 \sim N(\mu_1, \sigma_1^2) \]
\[ X_2 \sim N(\mu_2, \sigma_2^2) \]
\[ \Rightarrow p(X_1) \cdot p(X_2) \sim N \left( \frac{\sigma_1^2 \mu_1 + \sigma_2^2 \mu_2}{\sigma_1^2 + \sigma_2^2}, \frac{1}{\sigma_1^2 + \sigma_2^2} \right) \]

Multivariate Gaussians

\[ X \sim N(\mu, \Sigma) \]
\[ Y = AX + B \]
\[ \Rightarrow Y \sim N(A\mu + B, A\Sigma A^T) \]

\[ X_1 \sim N(\mu_1, \Sigma_1) \]
\[ X_2 \sim N(\mu_2, \Sigma_2) \]
\[ \Rightarrow p(X_1) \cdot p(X_2) \sim N \left( \frac{\Sigma_2 \Sigma_2 / \Sigma_1 + \Sigma_1 \Sigma_2 / \Sigma_1}{\Sigma_1 + \Sigma_2}, \frac{1}{\Sigma_1 + \Sigma_2} \right) \]

- We stay in the “Gaussian world” as long as we start with Gaussians and perform only linear transformations.
Discrete Kalman Filter

Estimates the state $x$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t$$

with a measurement

$$z_t = C_t x_t + \delta_t$$

Components of a Kalman Filter

- $A_t$: Matrix (nxn) that describes how the state evolves from $t$ to $t-1$ without controls or noise.
- $B_t$: Matrix (nxl) that describes how the control $u_t$ changes the state from $t$ to $t-1$.
- $C_t$: Matrix (kxn) that describes how to map the state $x_t$ to an observation $z_t$.
- $\epsilon_t$: Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance $R_t$ and $Q_t$ respectively.

Kalman Filter Updates in 1D

$$\text{be}(x_t) = \begin{cases} \mu_t = \overline{x}_t + K_t (z_t - \overline{z}_t) \\
\sigma_t^2 = (1 - K_t) \sigma_t^2 
\end{cases}$$

with

$$K_t = \frac{\sigma_t^2}{\sigma_t^2 + \sigma_{\epsilon t}^2}$$

$$\Sigma_t = (I - K_t) \Sigma_t$$

with

$$K_t = \Sigma_t C_t^T (C_t \Sigma_t C_t^T + Q_t)^{-1}$$
Kalman Filter Updates in 1D

\[
\begin{align*}
\overline{Bel}(x_t) &= \mu_t = a_t \mu_{t-1} + b_t u_t \\
\sigma_t^2 &= a_t^2 \sigma_{t-1}^2 + \sigma_{t,t-1}^2 \\
\overline{Bel}(x_t) &= \mu_t = A_t \mu_{t-1} + B_t u_t \\
\Sigma_t &= A_t \Sigma_{t-1} A_t^T + R_t
\end{align*}
\]

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

\[ Bel(x_0) = N(x_0; \mu_0, \Sigma_0) \]

Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and control plus additive noise:

\[
\begin{align*}
x_t &= A_t x_{t-1} + B_t u_t + \epsilon_t \\
p(x_t | u_t, x_{t-1}) &= N(x_t; A_t x_{t-1} + B_t u_t, R_t) \\
\overline{Bel}(x_t) &= \int p(x_t | u_t, x_{t-1}) \ overline{Bel}(x_{t-1}) \ dx_{t-1} \\
&\sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) \\
&\sim N(x_t; A_t x_{t-1} + B_t u_t, \Sigma_t)
\end{align*}
\]
• We stay in the “Gaussian world” as long as we start with Gaussians and perform only linear transformations.

Recall: Multivariate Gaussians

\[
\begin{align*}
X & \sim N(\mu, \Sigma) \\
Y &= AX + B \\
\Rightarrow \quad Y & \sim N(A\mu + B, A\Sigma A^T)
\end{align*}
\]

\[
\begin{align*}
X_i & \sim N(\mu_i, \Sigma_i) \\
X_j & \sim N(\mu_j, \Sigma_j) \\
\Rightarrow \quad p(X_i, X_j) & = N\left(\frac{\Sigma_i + \Sigma_j + \Sigma - \Sigma_i - \Sigma_j - \mu_i - \mu_j}{\Sigma_i + \Sigma_j + \Sigma - \Sigma_i - \Sigma_j - \mu_i - \mu_j}, \frac{1}{\Sigma_i + \Sigma_j + \Sigma - \Sigma_i - \Sigma_j - \mu_i - \mu_j}\right)
\end{align*}
\]

Linear Gaussian Systems: Dynamics

\[
\overline{p}(x_t) = \int p(x_t | u_{t-1}, x_{t-1}) \overline{p}(x_{t-1}) \, dx_{t-1}
\]

\[
\sim N(x_t; A_t x_{t-1} + B_t \mu_t, R_t) \sim N(x_t; \mu_{t-1}, \Sigma_{t-1})
\]

\[
\overline{p}(x_t) = \eta \exp \left\{ -\frac{1}{2} (x_t - A_t x_{t-1} - B_t \mu_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t \mu_t) \right\}
\exp \left\{ -\frac{1}{2} (x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1}) \right\} \, dx_{t-1}
\]

\[
\overline{p}(x_t) = \left[ \mu_t = A_t \mu_{t-1} + B_t \mu_t, \Sigma_t = A_t \Sigma_{t-1} A_t^T + R_t \right]
\]

Linear Gaussian Systems: Observations

• Observations are linear function of state plus additive noise:

\[
z_t = C_t x_t + \delta_t
\]

\[
p(z_t | x_t) = N(z_t; C_t x_t, Q_t)
\]

\[
\overline{p}(x_t) = \eta \quad p(z_t | x_t) \quad \overline{p}(x_t) \quad \downarrow \quad \downarrow
\]

\[
\sim N(z_t; C_t x_t, Q_t) \quad \sim N(x_t; \mu_t, \Sigma_t)
\]

Linear Gaussian Systems: Observations

\[
\overline{p}(x_t) = \eta \quad p(z_t | x_t) \quad \overline{p}(x_t) \quad \downarrow \quad \downarrow
\]

\[
\sim N(z_t; C_t x_t, Q_t) \quad \sim N(x_t; \mu_t, \Sigma_t)
\]

\[
\overline{p}(x_t) = \eta \exp \left\{ -\frac{1}{2} (z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t) \right\} \exp \left\{ -\frac{1}{2} (x_t - \mu_t)^T \Sigma_t^{-1} (x_t - \mu_t) \right\}
\]

\[
\overline{p}(x_t) = \left[ \mu_t = \mu_t + K_t (z_t - C_t \mu_t), \Sigma_t = (I - K_t C_t) \Sigma_t \right]
\]

\[
K_t = C_t \Sigma_t (C_t \Sigma_t + Q_t)^{-1}
\]
Kalman Filter Algorithm

1. Algorithm Kalman_filter( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\overline{\mu}_t = A \mu_{t-1} + B u_t$
4. $\overline{\Sigma}_t = A \Sigma_{t-1} A^T + R$

5. Correction:
6. $K_t = \overline{\Sigma}_t C^T (C \overline{\Sigma}_t C^T + Q)^{-1}$
7. $\mu_t = \overline{\mu}_t + K_t (z_t - C \overline{\mu}_t)$
8. $\Sigma_t = (I - K_t C) \overline{\Sigma}_t$

9. Return $\mu_t, \Sigma_t$